Technical Comments

A Numerical Method for a Stefan Problem

HAROLD J. BREAUX*
U.S. Army Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

Nomenclature†

L = thickness of slab or radius of cylinder or sphere
n = 0, 1, 2 for planar, cylindrical, or spherical symmetry
u = temperature at time t
U_m = melting, ablating, or subliming temperature
x = space coordinate
y = transformed space coordinate
z = receded distance
$\dot{z}$ = recession rate
Δx = increment in x
Δy = increment in y
σ = diffusivity

Introduction

In Ref. 1 Eppes presented a finite-difference heat conduction method applicable during surface recession. The method presented was restricted to problems meeting the following restrictions: 1) The method was limited to a homogeneous flat plate where the receding surface is located between the original position of the heated surface and the first interior nodal point. 2) The recession rate was either known from applicable experimental data or calculable by analytical methods. 3) The temperature of the receding surface remained constant during the recession process and the material thermal properties did not vary with temperature.

Despite the simplifications made possible by these assumptions, difficulties still arose. Eppes found it necessary, for example, to treat two surface conditions. The first surface condition was defined by $0 < z \leq \Delta x/2$ and the second by $\Delta x/2 < z \leq \Delta x$. This difficulty has been observed by Ehrlich,[2] who found it necessary to provide for four separate cases. The additional cases arise when the boundary is allowed to intersect a grid line in situations more general than that allowed in restriction 1. The greatest shortcoming of Eppes' method, however, is the difficulty associated with adapting his technique to cases of more general boundary conditions than that allowed in (2) and (3). A more general problem is one having the recession rate $\dot{z}$ governed by a differential equation involving $u(z,t)$, $u_x(z,t)$, z, and t. In this general case the conventional explicit and implicit finite-difference methods require iteration for locating the position of the boundary. This is not only time consuming, but when the several cases must be anticipated, the programming details make these procedures unwieldly.

The method described by the author overcomes these difficulties in addition to having certain features which provide for easy adjustment of step size in the time variable. This makes possible the design of more efficient computer programs. The method consists of applying the "method of lines" to a transformed partial differential equation and transformed boundary conditions. This transformation, suggested by Landau,[3] serves to fix the boundary in a new space coordinate. Similar transformations have been used by Crank[4] and by Murray and Landis.[5]

Received October 27, 1967.
* Research Mathematician, Computing Laboratory.
† Subscripts t, x, y denote partial derivatives; integer subscripts and subscripts containing j denote temperature along a line, e.g., u_j is equal to temperature along the line $y = j\Delta y$ at time t.

Statement of the Problem

The equations governing the process of ablation, sublimation or melting with melt removed involve a moving boundary and are generally called "Stefan-like" problems. Typically these equations take the form

$$u_t = \sigma[u_{xx} + (n/x)u_x] \qquad z < x < L \qquad t > 0 \quad (1)$$

$$u_x = f(u,t) \qquad x = L \qquad t > 0 \quad (2)$$

$$u = U_m \qquad x = z \qquad t > 0 \quad (3)$$

$$\dot{z} = g(u,u_x,z,t) \qquad x = z \qquad t > 0 \quad (4)$$

$$u = h(x) \qquad 0 \leq x \leq L \qquad t = 0 \quad (5)$$

The form of Eq. (1) allows the use of slab, cylindrical, or spherical geometry by setting $n = 0,1,2$, respectively. f, g, and h have been left arbitrary since the method presented places no restriction on their functional nature.

Landau's transformation $y = (L - x)/(L - z)$ fixes the boundaries $x = z$ at $y = 1$ and $x = L$ at $y = 0$. The problem in the new space coordinate is specified by the equations

$$u_t = au_{yy} - bu_y \qquad 0 < y < 1 \qquad t > 0 \quad (6)$$

$$u_y = -(L - z)f(u,t) \qquad y = 0 \qquad t > 0 \quad (7)$$

$$u = U_m \qquad y = 1 \qquad t > 0 \quad (8)$$

$$\dot{z} = g(u,u_y,z,t) \qquad y = 1 \qquad t > 0 \quad (9)$$

$$u = h(y) \qquad 0 \leq y \leq 1 \qquad t = 0 \quad (10)$$

where

$$a = \sigma(L - z)^{-2} \quad (11)$$

and

$$b = \sigma n(L - z)^{-1}[L - y(L - z)]^{-1} + y\dot{z}(L - z)^{-1} \quad (12)$$

Numerical Solution

In the method of lines the yt plane is divided into $(J + 1)$ subdivisions by the lines $y = j\Delta y$, $j = 0,1, \ldots, J$. The partial derivatives u_y and u_{yy} are replaced with appropriate difference formulas and u_t is replaced with (du_j/dt), i.e., the time derivative of u along the jth line. For difference formulas of accuracy $0(\Delta y^2)$, Eq. (6) becomes

$$(du_j/dt) = a(u_{j+1} - 2u_j + u_{j-1})/(\Delta y)^2 - b(u_{j+1} - u_{j-1})/2\Delta y \qquad j = 1,2, \ldots, J - 1 \quad (13)$$

where a and b are evaluated at the appropriate y and z. An effective method, consistent with accuracy $0(\Delta y^2)$, for handling the boundary condition (7) is that listed by Fox,[6] p. 248. The difference equation (13) is assumed to hold for $j = 0$. The temperature u_{-1} exterior to the field of interest is eliminated by the difference formula

$$u_{-1} = u_1 - 2\Delta y(L - z)f(u_0,t) \quad (14)$$

obtained from the boundary condition (7). In the equation for the recession rate (9), u_y is replaced with

$$u_y \approx (u_{J-2} - 4u_{J-1} + 3u_J)/2\Delta y \quad (15)$$

an approximation also of order $0(\Delta y^2)$.

The problem has now been reduced to solving $(J + 1)$ ordinary differential equations. $(J - 1)$ equations arise from (13), with an additional equation arising from (13) for $j = 0$ when u_{-1} is obtained from (14). The remaining equation is that for $\dot{z}$, Eq. (9).

Most computing machines are equipped with powerful subprograms for solving systems of ordinary differential equations. These programs are usually designed to select automatically the interval of integration based on criteria designed to bound the truncation error. Since the problems under consideration are often characterized by rapidly decaying transients, it follows that these problems allow vastly differing integration intervals to maintain a fixed truncation error. Explicit finite-difference methods require adjusting the space increment when varying the time increment to maintain a fixed stability ratio. For this reason, and because of a lack of criteria for adjusting step size in both explicit and implicit difference schemes, the method of lines seems to be the simplest, most efficient, and most flexible method for the type of problem presented herein. The greatest advantage, however, is its ability to find the position of the moving boundary without iteration. Variations of the procedure described in this paper have been used successfully by the author on problems involving the recession of a burning fluid and the solidification of explosives. In addition, the method can easily be extended to heat flow problems involving composite materials or to materials having thermal properties dependent on temperature.

References

[1] Eppes, R., Jr., "A Finite-Difference Heat Conduction Method Applicable During Surface Recession," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1679–1682.

[2] Ehrlich, L. W., "A Numerical Method of Solving a Heat Flow Problem With Moving Boundary," *Journal of the Association for Computing Machinery*, Vol. 5, 1958, pp. 161–176.

[3] Landau, H. G., "Heat Conduction In a Melting Solid," *Quarterly of Applied Mathematics*, Vol. 8, 1950, pp. 81–94.

[4] Crank, J., "Two Methods for the Numerical Solution of Moving-Boundary Problems in Diffusion and Heat Flow," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 10, Pt. 2, 1957, pp. 220–231.

[5] Murray, W. D. and Landis, F., "Numerical and Machine Solutions of Transient Heat-Conduction Problems Involving Melting or Freezing," *Transactions of the American Society of Mechanical Engineers*, Vol. 81, 1959, pp. 106–112.

[6] Fox, L., *Numerical Solution of Ordinary and Partial Differential Equations*, Addison-Wesley, Reading, Mass., 1962, pp. 248–249.

Reply by Author to H. J. Breaux

RICHARD EPPES JR.*
U.S. Army Missile Command, Redstone Arsenal, Ala.

IN his comment[1] on Ref. 2, Breaux stated that the method presented was restricted by several conditions. The method in itself is not restricted, as claimed by Breaux; only the presentation in the Note was restricted for purposes of brevity.

A footnote[2] in my Note stated that the method has been extended to encompass radial heat flow in cylinders and spheres, finite-difference methods for inward and outward surface recession, equations describing cylindrical and spherical sublimation-conduction for several typical composite material arrangements, equations defining criteria for stopping surface recession, and equations to determine heat flow after recession terminates. These techniques all have been used successfully in the past and no shortcomings nor difficulties have been encountered.

As stated in Ref. 2, the equations presented are applicable only to the first interior nodal point where this point is located less than one Δ space increment from the receding surface. The surface temperature is constant (variable with time if one desires) and the other nodal points, with the exception of the second nodal point, are calculated using general explicit finite-difference equations. Once the recession front reaches the original first interior nodal point (T_2), then the equations presented[2] are applied to the original third nodal point (T_3), as stated on p. 1680[2] after Eq. (3). This procedure of handling the first interior nodal point temperature is continued until the receding surface is within one Δ increment of the back surface or an interface as stated in the Note[2] on p. 1680. Special equations, of no greater complexity, must then be applied.

The reason that the presentation was restricted to cases where the temperature of the receding surface remained constant during the recession process and the material thermal properties did not vary with temperature was to allow for comparisons of data to exact solution results. Variable thermal property data have been used and the receding surface temperature varying with time can be very easily incorporated.

To circumvent compressing the grid and encountering small time increments, especially as the receding surface approaches a backside or substructure interface, the nonshifting grid technique was considered advantageous. Inherent to a nonshifting grid is computational efficiency and a fixed stability ratio. If the computed temperatures are to be compared with empirical thermocouple data from a receding material, the fixed grid is most desirable and simple to arrange.

As stated in the Note,[2] the procedure requires a negligible increase in computer time over an identical nonrecession case, primarily because all calculations internal to the first interior node are made by ordinary explicit finite differences. The computer program efficiency is thus comparable to a nonrecession explicit finite difference program. The efficiency has been verified by past experience in using the program.

This recession-conduction procedure is extremely simple for extending general explicit finite-difference heat conduction routines to provide analytical capability in calculating the temperature of a structure during surface recession.

References

[1] Breaux, H. J., "A Numerical Method for a Stefan Problem," *AIAA Journal*, Vol. 6, No. 9, Sept. 1968, pp. 1821–1822.

[2] Eppes, R., Jr., "A Finite-Difference Heat Conduction Method Applicable During Surface Recession," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1679–1682.

Received April 1, 1968.

* Research Engineer, Structures and Mechanics Laboratory, Research and Development Directorate. Member AIAA.

Comment on "Inner Region of Transpired Turbulent Boundary Layers"

THOMAS J. DAHM* AND ROBERT M. KENDALL†
Aerotherm Corporation, Palo Alto, Calif.

IN a recent Note Stevenson[1] discusses the disparity between the theoretical friction factors for the transpired turbulent boundary layer predicted in Ref. 2 and those in Ref. 3. The

Received April 15, 1968.

* Manager, Fluid Dynamics Department. Member AIAA.

† Manager, Analytical Services Division. Member AIAA.